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A NOTE ON THE DECOMPOSITION OF ELASTOPLASTIC FINITE DEFORMATIONS

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Abstract—A new derivation and interpretation of the decomposition recently introduced by Nemat-Nasser are presented.

In the finite elastoplastic deformation theory many authors, e.g. Lee[1], Lehmann[2, 3], Sidoroff[4], etc. decompose the deformation gradient multiplicatively into the plastic and elastic parts. Recently, Nemat-Nasser[5] has proposed an additive decomposition which, not like the multiplicative decomposition, leads to a stretch rate decomposition where the elastic stretch rate does not depend of the plastic deformation. Here we give an alternative derivation and interpretation of Nemat-Nasser's decomposition.

In the infinitesimal elastoplastic deformation theory the strain tensor is additively divided into the elastic and plastic constituents: $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p$. The material differentiation of this formula leads to the additive decomposition of strain rate tensor: $\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p$. $\boldsymbol{\epsilon}^e$ represents such a strain state to which the deformed body in an elastic unloading process should be inversely subjected that all material elements become unstressed. Generally in nonhomogeneous deformations such an unstressed compatible state can not be obtained. For the purpose of material property studies, however, it is sufficient to consider homogeneous deformations where all material elements undergo the same deformation and stress changes. The unstressed state is then obtained by removal of the surface traction. For homogeneous deformations, $\boldsymbol{\epsilon}^e$ and $\boldsymbol{\epsilon}^p$ are constant and the compatibility conditions are identically fulfilled. By prescribing the displacement and rotation (or velocity and rotation velocity) at some point we can by means of line integration calculate at arbitrary point the displacement

$$\mathbf{u} = \mathbf{u}^e + \mathbf{u}^p \tag{1}$$

or the velocity

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}^e + \dot{\mathbf{u}}^p. \tag{2}$$

This means that the displacement (or velocity) vector is a sum of two imagined vectors: $\mathbf{u}^e$ represents the elastic part and $\mathbf{u}^p$ the plastic one. These two constituents are uncoupled.

If one treats the finite elastoplastic deformation in the uncoupled theory as a process consisting of an infinite series of infinitesimal uncoupled elastoplastic deformations, then formula (2) retains its validity for every moment. A material time integration of it leads to the same formula for the displacement (1), this time already in finite deformations. The interpretation is the same as in the infinitesimal theory. If the position vectors of a typical point in the initial and actual configurations are denoted by $\mathbf{X}$ and $\mathbf{x}$, respectively, then (1) can be rewritten as

$$\mathbf{x} - \mathbf{X} = \underset{(e)}{\mathbf{x}} - \mathbf{X} + \underset{(p)}{\mathbf{x}} - \mathbf{X}, \tag{3}$$

or

$$\mathbf{x} = \underset{(e)}{\mathbf{x}} + \underset{(p)}{\mathbf{x}} - \mathbf{X}. \tag{4}$$

$\underset{(e)}{\mathbf{x}}$ ($\underset{(p)}{\mathbf{x}}$) is the position which the typical point would have occupied in an imagined purely elastic (plastic) deformation. From (4) we obtain straight the additive decomposition of the deformation gradient:

$$\mathbf{F} \equiv \operatorname{grad} \mathbf{x} = \operatorname{grad} \underset{(e)}{\mathbf{x}} - \operatorname{grad} \underset{(p)}{\mathbf{x}} - \mathbf{I} \equiv \underset{(e)}{\mathbf{F}} + \underset{(p)}{\mathbf{F}} - \mathbf{I}, \tag{5}$$

where $\mathbf{I}$ is the identity tensor. This decomposition has been recently introduced by Nemat–Nasser[5] in a different way. A material element $d\mathbf{X}$ is deformed by mapping $\mathbf{F}$ into

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \tag{6}$$

in the total deformation. Writing $\mathbf{F}$ in form

$$\mathbf{F} = (\mathbf{I} + \underset{(p)}{\mathbf{F}}\underset{(e)}{\mathbf{F}}^{-1} - \underset{(e)}{\mathbf{F}}^{-1})\underset{(e)}{\mathbf{F}} \tag{7}$$

or

$$\mathbf{F} = (\mathbf{I} + \underset{(e)}{\mathbf{F}}\underset{(p)}{\mathbf{F}}^{-1} - \underset{(p)}{\mathbf{F}}^{-1})\underset{(p)}{\mathbf{F}}, \tag{8}$$

one can imagine that $d\mathbf{X}$ is deformed elastically or plastically into

$$d\underset{(e)}{\mathbf{x}} = \underset{(e)}{\mathbf{F}}\, d\mathbf{X} \tag{9}$$

or

$$d\underset{(p)}{\mathbf{x}} = \underset{(p)}{\mathbf{F}}\, d\mathbf{X}, \tag{10}$$

respectively. Similarly to the Green's deformation tensor or the Lagrangian strain tensor

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \tag{11}$$

in the total deformation, one can define the analogous tensors for ealstic and plastic parts:

$$\underset{(e)}{\mathbf{C}} = \underset{(e)}{\mathbf{F}}^T \underset{(e)}{\mathbf{F}}, \qquad \underset{(e)}{\mathbf{E}} = \frac{1}{2}(\underset{(e)}{\mathbf{C}} - \mathbf{I}) \tag{12}$$

and

$$\underset{(p)}{\mathbf{C}} = \underset{(p)}{\mathbf{F}}^T \underset{(p)}{\mathbf{F}}, \quad \underset{(p)}{\mathbf{E}} = \frac{1}{2}(\underset{(p)}{\mathbf{C}} - \mathbf{I}), \tag{13}$$

respectively. Thus, we have the elastic stretch-squared

$$\underset{(e)}{\Lambda}^2 = \frac{|\underset{(e)}{\mathbf{F}}\, d\mathbf{X}|^2}{d\mathbf{X}^2} = \mathbf{N} \cdot \underset{(e)}{\mathbf{C}}\mathbf{N} \tag{14}$$

and the plastic stretch-squared

$$\underset{(p)}{\Lambda}^2 = \frac{|\underset{(p)}{\mathbf{F}}\, d\mathbf{X}|^2}{d\mathbf{X}^2} = \mathbf{N} \cdot \underset{(p)}{\mathbf{C}}\mathbf{N} \tag{15}$$

where $\mathbf{N}$ is the unit vector.

If one takes the material time derivative of both sides of (5), one obtains

$$\dot{\mathbf{F}} = \underset{(e)}{\dot{\mathbf{F}}} + \underset{(p)}{\dot{\mathbf{F}}}, \tag{16}$$

which, upon multiplication by $\mathbf{F}^{-1}$, yields unambiguously the decomposition of velocity gradient and strain rate tensor:

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \underset{(e)}{\dot{\mathbf{F}}\mathbf{F}^{-1}} + \underset{(p)}{\dot{\mathbf{F}}\mathbf{F}^{-1}} \equiv \underset{(e)}{\mathbf{L}} + \underset{(p)}{\mathbf{L}}, \tag{17}$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \frac{1}{2}[(\underset{(e)}{\mathbf{L}} + \underset{(e)}{\mathbf{L}^T}) + (\underset{(p)}{\mathbf{L}} + \underset{(p)}{\mathbf{L}^T})] \equiv \underset{(e)}{\mathbf{D}} + \underset{(p)}{\mathbf{D}}. \tag{18}$$

Thus, the rate of the work $\dot{w}$ per unit mass and its decomposition are expressed by

$$\dot{w} = \underset{(e)}{\dot{w}} + \underset{(p)}{\dot{w}} = \frac{1}{\rho}\mathbf{t} : \mathbf{D}, \tag{19}$$

$$\underset{(e)}{\dot{w}} = \frac{1}{\rho}\mathbf{t} : \underset{(e)}{\mathbf{D}}, \tag{20}$$

$$\underset{(p)}{\dot{w}} = \frac{1}{\rho}\mathbf{t} : \underset{(p)}{\mathbf{D}}, \tag{21}$$

t and ρ being the Cauchy stress tensor and current mass density, respectively.

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REFERENCES

1. E. H. Lee, Elastic–plastic deformation at finite strain. *J. Appl. Mech.* **36**, 1–6 (1969).
2. Th. Lehmann, On large elastic–plastic deformations. *Proc. Int. Symp. on Foundations of Plasticity, Warsaw* 1972. Noordhoff, Amsterdam (1974).
3. Th. Lehmann, Some aspects of non-isothermic large inelastic deformations. *SM Archives* 3, 261–317 (1978).
4. F. Sidoroff, The geometrical concept of intermediate configuration and elastic–plastic finite strain. Arch. Mech. **25**, 299–308 (1973).
5. S. Nemat-Nasser, Decomposition of strain measures and their rates in finite deformation elastoplasticity. *Int. J. Solids Structures* **15**, 155–166 (1979).